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APPLICATION OF DUAL INTEGRAL EQUATIONS TO THE PROBLEM OF TORSION OF AN ELASTIC SPACE, WEAKENED BY A CONICAL CRACK OF FINITE DIMENSIONS

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We examine the axisymmetric problem of the torsion of an elastic space, weakened by a conical crack, under the assumption that on the boundaries of the crack the tangential displacements or the shear stresses are prescribed. The solution is obtained by the application of dual integral equations related to the Mellin integral transform, similar to those studied in [1]. The result is represented in quadratures in terms of auxiliary functions which are solutions of a Fredholm equation of the second kind. As an example, we solve the problem of torsion of the space with a rigid conical inclusion.

1. Formulation of the problem. We consider the axisymmetric problem of the torsion of an elastic space, weakened by a conical crack. Let r, ϑ, φ be the system of spherical coordinates, whose origin coincides with the vertex of the crack, while the z-axis is the axis of symmetry (Fig. 1). For this choice of coordinates, the problem under consideration reduces to the determination of the only nonzero component $u_\varphi = u(r, \vartheta)$ of the displacement vector, satisfying the equation

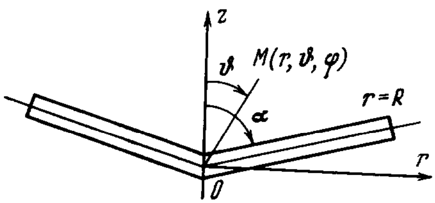

Fig. 1

$$\Delta u - \frac{u}{r^2 \sin^2 \vartheta} = 0 \qquad (1.1)$$

The components of the stress tensor can be expressed in terms of u with the relations

$$\tau_{\vartheta\varphi} = \mu \left[\frac{\partial}{\partial \vartheta} \left(\frac{u}{\sin \vartheta} \right) \right] \frac{\sin \vartheta}{r}, \qquad \tau_{r\varphi} = \mu \left[\frac{\partial u}{\partial r} - \frac{u}{r} \right] \qquad (1.2)$$

where μ is the shear modulus. The desired function must satisfy boundary conditions of one of the following two types (for given tangential displacements or shear stresses):

$$u\,|_{\vartheta=\alpha} = f(r), \qquad 0 \leqslant r < R \qquad (1.3)$$

$$\sin \vartheta \frac{\partial}{\partial \vartheta} \left(\frac{u}{\sin \vartheta} \right) \Big|_{\vartheta=\alpha} = \frac{1}{\mu} g(r), \quad 0 \leqslant r < R \qquad (1.4)$$

Here $f(r)$ and $g(r)$ are specified continuous functions. In addition, the conditions at infinity and the conditions of the behavior of the desired function near the vertex of the cone must be satisfied

$$u\,|_{r\to\infty} = O(r^{-1}), \quad \tau_{r\varphi} = O(r^{-2}), \quad \tau_{\vartheta\varphi} = O(r^{-2}) \qquad (1.5)$$
$$u\,|_{\vartheta=0,\ r\to 0} = O(1); \qquad \tau_{r\varphi} = O(r^{-\delta}),\ 0 < \delta < 1$$

$$\tau_{\vartheta\varphi} = O(r^{-\varepsilon}), \qquad 0 < \varepsilon < 1 \tag{1.6}$$

2. The reduction of the problem to dual integral equations.

For the solution of the formulated problem we will make use of the method of separation of variables and we will seek the solution in the form

$$u = \sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} M(\tau) \frac{P^1_{-1/2+i\tau}(\cos\vartheta)}{P^1_{-1/2+i\tau}(\cos\alpha)} e^{-i\tau \ln r/R} d\tau, \qquad 0 \leqslant \vartheta < \alpha$$

$$u = \sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} M(\tau) \frac{P^1_{-1/2+i\tau}(-\cos\vartheta)}{P^1_{-1/2+i\tau}(-\cos\alpha)} e^{-i\tau \ln r/R} d\tau, \qquad \alpha < \vartheta < \pi \tag{2.1}$$

for the case of boundary conditions (1.3) and in the form

$$u = \sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} N(\tau) \frac{\sin\alpha\, P^1_{-1/2+i\tau}(\cos\vartheta)}{P^2_{-1/2+i\tau}(\cos\alpha)} e^{-i\tau \ln r/R} d\tau, \qquad 0 \leqslant \vartheta < \alpha$$

$$u = \sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} N(\tau) \frac{-\sin\alpha\, P^1_{-1/2+i\tau}(-\cos\vartheta)}{P^2_{-1/2+i\tau}(-\cos\alpha)} e^{-i\tau \ln r/R} d\tau, \qquad \alpha < \vartheta < \pi \tag{2.2}$$

in the case of the boundary conditions (1.4). In these expressions $M(\tau)$ and $N(\tau)$ are continuous functions, subject to determination, $P_\nu^{\,m}(x)$ is the associated spherical Legendre function of the first kind. The expressions (2.1) and (2.2) satisfy formally the differential equation (1.1), the conditions (1.5) and (1.6) and also the continuity conditions at the symmetry axis and at the surface $\vartheta = \alpha$.

The boundary conditions (1.3) and (1.4) and the continuity requirements of the normal derivative of the function at the crossing of the surface $(R < r < \infty,\ \vartheta = \alpha)$, lead us to the dual equations for the determination of the functions $M(\tau)$ and $N(\tau)$. These equations have the form

$$\sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} M(\tau) e^{-i\tau \ln r/R} d\tau = f(r), \qquad r < R \tag{2.3}$$

$$\sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} M(\tau) \frac{(1/2 + 2\tau^2)\operatorname{ch}\pi\tau}{\sin\alpha\, P^1_{-1/2+i\tau}(\cos\alpha)\, P^1_{-1/2+i\tau}(-\cos\alpha)} e^{-i\tau \ln r/R} d\tau = 0, \ r < R$$

in the problem with the prescribed displacements and

$$\sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} N(\tau) e^{-i\tau \ln r/R} d\tau = \frac{g(r)}{\mu}, \qquad r < R$$

$$\sqrt{\frac{R}{r}} \int_{-\infty}^{\infty} N(\tau) \frac{(1/2 + 2\tau^2)\operatorname{ch}\pi\tau}{\pi P^2_{-1/2+i\tau}(\cos\alpha)\, P^2_{-1/2+i\tau}(-\cos\alpha)} e^{-i\tau \ln r/R} d\tau = 0, \quad r > R \tag{2.4}$$

in the problem with prescribed shear stresses. The equations (2.3) and (2.4) belong to the class of dual integral equations connected with the Mellin transform, investigated in [1]. Let us prove that the solution of these equations can be expressed by quadratures in terms of auxiliary functions which satisfy Fredholm integral equations of the second kind with symmetric kernels. The latter can be effectively solved and they allow us to obtain convenient computational formulas for the components of the stress tensor and for other quantities which present interest.

3. The solution of the dual integral equations. We consider the dual integral equations (2.3). We will seek their solution in the form

$$M(\tau)=\frac{\sin\alpha P^1_{-1/2+i\tau}(\cos\alpha)\,P^1_{-1/2+i\tau}(-\cos\alpha)}{4\sqrt{\pi R}\,(1+4\tau^2)\operatorname{ch}\pi\tau}\times$$
$$\frac{\Gamma(5/4+1/2\,i\tau)}{\Gamma(3/4+1/2\,i\tau)}\int_0^R\frac{\varphi(t)}{\sqrt{t}}\,e^{i\tau\ln t/R}\,dt \qquad (3.1)$$

Here $\varphi(t)$ is an unknown function, continuous together with its derivative in the interval $(0, R)$ and satisfying the condition $\varphi(t)\sqrt{t}\to 0$ when $t\to 0$, while $\Gamma(z)$ is the Euler gamma function. If we substitute (3.1) into the second of the equations (2.3) and if we take into account the known relation

$$\frac{1}{4\sqrt{\pi}}\sqrt{\frac{R}{r}}\int_{-\infty}^{\infty}\frac{\Gamma(1/4+1/2\,i\tau)}{\Gamma(3/4+1/2\,i\tau)}\,e^{-i\tau\ln r/R}\,d\tau=\begin{cases}\dfrac{R}{\sqrt{R^2-r^2}}, & r<R\\ 0, & r>R\end{cases}$$

then it is easy to see that the equation under consideration is identically satisfied. The substitution of (3.1) into the remaining equation leads us to a Fredholm integral equation of the second kind

$$\psi(x)=l(x)-\int_0^\infty G(|x-y|)\,\psi(y)\,dy,\qquad 0<x<\infty \qquad (3.2)$$

Here

$$t=Re^{-x},\quad s=Re^{-y},\quad e^{-1/2\,x}\varphi(Re^{-x})=\psi(x)$$
$$l(x)=-16e^{3/2\,x}\frac{d}{dx}\int_x^\infty\frac{f(Re^{-y})\,e^{-3y}}{(e^{-2x}-e^{-2y})^{1/2}}\,dy$$
$$G(x)=\frac{1}{\pi}\int_0^\infty\left\{\frac{\sin\alpha\,P^1_{-1/2+i\tau}(\cos\alpha)\,P^1_{-1/2+i\tau}(-\cos\alpha)}{[P^1_{-1/2+i\tau}(0)]^2}-1\right\}\cos\tau x\,d\tau$$

This can be easily seen if we perform the computations, similar to those in [1], and if we make use of the discontinuous integral

$$\frac{1}{4\sqrt{\pi}}\sqrt{\frac{R}{r}}\int_{-\infty}^{\infty}\frac{\Gamma(3/4-1/2\,i\tau)}{\Gamma(5/4-1/2\,i\tau)}\,e^{-i\tau\ln r/R}\,d\tau=\begin{cases}0, & r<R\\ \dfrac{R^2}{r\sqrt{r^2-R^2}}, & r>R\end{cases}$$

For Eqs. (2.4) we seek the solution in the form

$$N(\tau)=\frac{(1/2-i\tau)\,P^2_{-1/2+i\tau}(\cos\alpha)\,P^2_{-1/2+i\tau}(-\cos\alpha)}{2\sqrt{\pi R}\,(1/4+\tau^2)^2\operatorname{ch}\pi\tau}\,\frac{\Gamma(3/4+1/2\,i\tau)}{\Gamma(1/4+1/2\,i\tau)}\int_0^R\frac{\varphi(t)}{\sqrt{t}}\,e^{i\tau\ln t/R}\,dt \qquad (3.3)$$

In a manner similar to the previous one, we arrive at Eq. (3.2), but here

$$l(x)=-\frac{1}{2\mu}\,e^{1/2\,x}\int_x^\infty\frac{g(Re^{-y})\,e^{-2y}}{(e^{-2x}-e^{-2y})^{1/2}}\,dy$$

$$G(x) = \frac{1}{\pi}\int_0^\infty \left\{\frac{\sin\alpha P^2_{-1/2+i\tau}(\cos\alpha)\, P^2_{-1/2+i\tau}(-\cos\alpha)}{[P^2_{-1/2+i\tau}(0)]^2} - 1\right\}\cos\tau x\, d\tau$$

The equation (3.2) belongs to the class which is solvable by the Wiener-Hopf method. However, the application of this method to the given equation or to the initial dual equations is connected with the factorization of complicated functions and leads to formulas which are not very suitable for numerical computations. Therefore, it is more rational to apply the iteration method as well as the expansion into a series of powers of a small parameter.

4. The solution of the problem for the case of a large apex angle of the cone. The method of the small parameter. We consider the method of solution of Eq. (3.2), based on the expansion of $\operatorname{ctg}^2\alpha$ in a power series, which leads to relatively simple formulas. The basis of the method is the expansion formula of the product of spherical functions into series

$$\sin\alpha\, P^m_{-1/2+i\tau}(\cos\alpha)\, P^m_{-1/2+i\tau}(-\cos\alpha) = \frac{1}{2^m\pi^2}\operatorname{ch}^2\pi\tau \times$$

$$\frac{\Gamma^2(1/2+i\tau+m)\,\Gamma^2(1/2-i\tau+m)}{\Gamma^2(1+m)}\sum_{k=0}^{\infty}(-1)^k\frac{(m+1/2)_k}{k!}\times$$

$${}_3F_2\left(\frac{1}{2}-k+m,\ \frac{1}{2}+i\tau+m,\ \frac{1}{2}-i\tau+m;\ 2m+1,\ m+1;\ 1\right)\operatorname{ctg}^{2k}\alpha \qquad (4.1)$$

where ${}_3F_2$ is the generalized hypergeometric function. The derivation of this formula is given in the Appendix.

Making use of (4.1), we can represent the kernel in the form

$$G(x) = \frac{1}{\pi}\int_0^\infty \left\{\frac{\sin\alpha P^m_{-1/2+i\tau}(\cos\alpha)\, P^m_{-1/2+i\tau}(-\cos\alpha)}{[P^m_{-1/2+i\tau}(0)]^2} - 1\right\}\times$$

$$\cos\tau x d\tau = \sum_{k=1}^{\infty} G_k(x)\operatorname{ctg}^{2k}\alpha \qquad (4.2)$$

$$G_k(x) = (-1)^k\frac{(m+1/2)_k}{k!}\int_0^\infty \frac{\omega_k(\tau)\,[P^m_{-1/2+i\tau}(0)]^2}{2^m\pi^2(m!)^2}\cos\tau x d\tau$$

$$\omega_k(\tau) = \frac{\Gamma^2(1/2+i\tau+m)\,\Gamma^2(1/2-i\tau+m)}{[P^m_{-1/2+i\tau}(0)]^2}\operatorname{ch}^2\pi\tau \times$$

$${}_3F_2\left(\frac{1}{2}-k+m,\ \frac{1}{2}+i\tau+m,\ \frac{1}{2}-i\tau+m;\ 2m+1,\ m+1;\ 1\right)$$

The series (4.2) converges rapidly for small values of $\operatorname{ctg}^2\alpha$, i.e. for α close to $1/2\,\pi$ (cone with large apex angle). The coefficients of the series can be obtained by numerical integration and the values of the functions $\omega_k(\tau)$ are determined by the formulas

$$\omega_0 = 2^m,\quad \omega_1 = \frac{8\Gamma^2(3/4+1/2m+1/2i\tau)}{\Gamma^2(1/4-1/2m+1/2i\tau)}\,\frac{\Gamma^2(3/4-1/2m+1/2i\tau)}{\Gamma^2(1/4+1/2m+1/2i\tau)} + \frac{1}{2} - 2\tau^2 - 2m^2$$

$$[(k+{}^1/_2m+{}^1/_2)^2+{}^1/_4m^2]\,\omega_{k+1}+[\tau^2-{}^1/_4-2k^2]\,\omega_k+[(k-{}^1/_2m-{}^1/_2)^2+{}^1/_4m^2]\,\omega_{k-1}=0 \tag{4.3}$$

The expansion of the kernel $G(x)$ of Eq. (3.2) in the problem with prescribed displacements and in the problem with prescribed shear stresses can be obtained from (4.1) as particular cases if we set $m = 1$, $m = 2$, respectively.

If we seek the solution of the considered integral equations in the form

$$\psi(x)=\sum_{k=0}^{\infty}\psi_k(x)\operatorname{ctg}^{2k}\alpha \tag{4.4}$$

then for the determination of the coefficients $\psi_k(x)$ we obtain a series of recursion relations

$$\psi_0(x)=l(x) \tag{4.5}$$

$$\psi_k(x)=-\sum_{n=1}^{k}\int_0^\infty G_n(|x-y|)\,\psi_{k-n}(y)\,dy, \qquad k=1,2\ldots$$

The application of this method to an actual problem is given below.

5. The torsion of the elastic space with a rigid conical inclusion. We consider the problem of the torsion of the elastic space with an inclusion in the form of a rigid thin cone (Fig. 1). We represent the tangential displacement in the form of a sum of displacements originating from the torsion of a homogeneous space and additional displacements induced by the presence of the cone

$$u_\varphi={}^1/_2\gamma r^2\sin 2\vartheta+u$$

where γ is the constant angle of twist per unit length. Then the problem reduces to the solution of the problem considered in Sect. 3, for

$$f(r)=\theta r-{}^1/_2\gamma r^2\sin 2\alpha$$

Here θ is the unknown angle of rotation of the cone. The value of this angle must be obtained from the condition of the vanishing of the moment of the shear stresses which act upon the cone.

Applying the method developed above, we arrive at the following integral equation:

$$\psi(x)=16\theta e^{-3/_2x}+3\pi\gamma R\sin 2\alpha e^{-5/_2x}-\int_0^\infty G(|x-y|)\,\psi(y)\,dy, \quad 0\leqslant x<\infty$$

where the kernel $G(x)$ is defined by the equality (4.2) for $m = 1$. It is convenient to represent the solution of the latter equation in the form

$$\psi(x)=64\theta\psi_1(x)-15\pi\gamma R\sin 2\alpha\psi_2(x)$$

Here $\psi_1(x)$ and $\psi_2(x)$ are the solutions of the equations

$$\psi_1(x)=4e^{-3/_2x}-\int_0^\infty G(|x-y|)\,\psi_1(y)\,dy$$

$$\psi_2(x)=5e^{-5/_2x}-\int_0^\infty G(|x-y|)\,\psi_2(y)\,dy$$

respectively. The functions $\psi_1(x)$ and $\psi_2(x)$ can be expressed in the form of the series

$$\psi_1(x) = \sum_{k=0}^{\infty} \psi_{k,1}(x) \operatorname{ctg}^{2k}\alpha$$

$$\psi_2(x) = \sum_{k=0}^{\infty} \psi_{k,2}(x) \operatorname{ctg}^{2k}\alpha$$

The computation of the coefficients of these series has been performed according to the scheme of Sect. 4, for which the computer BESM-4 has been used.

After determining the functions $\psi_1(x)$ and $\psi_2(x)$, the resulting moment applied to the cone is given by the formula

$$M = 3\pi^2 R^2 \mu \sin^2\alpha \int_0^\infty [640\theta\psi_1(x) - 15\pi\gamma R \sin^2\alpha\psi_2(x)]\, e^{-5/2x}\, dx$$

The requirement $M = 0$ leads now to the desired relation between the angle θ and the given twisting angle γ

$$\frac{\theta}{\gamma} k = \int_0^\infty \psi_2(x)\, e^{-5/2x}\, dx \Big/ \int_0^\infty \psi_1(x)\, e^{-5/2x}\, dx$$

$$k = 64\,(15\pi R \sin 2\alpha)^{-1}$$

For the values of α which are close to $^1/_2\,\pi$, we obtain a simple expression in the form of an expansion in powers of $\operatorname{ctg}^2\alpha$

$$\theta / \gamma k = 1 + 0.3578 \operatorname{ctg}^2\alpha + 0.0976 \operatorname{ctg}^4\alpha + \ldots$$

Appendix. We give a short proof of formula (4.1). We start with the integral representation (*)

$$P_\nu{}^m(\cos\alpha)\, P_\nu{}^m(-\cos\alpha) = (-1)^{m-1} \frac{2^m \Gamma(m + {}^1/_2)}{\pi\sqrt{\pi}} \times$$

$$\sin^{2m}\alpha \sin\pi\nu \int_1^\infty \frac{(t+1)^{1/2m} P_\nu{}^m(t)\, dt}{(t-1)^{1/2m+1/2}(t - \cos 2\alpha)^{m+1/2}}$$

$$(-1 < \operatorname{Re}\nu < 0)$$

Substituting here the expansion

$$\frac{(t+1)^{1/2m}(t - \cos 2\alpha)^{-m-1/2}}{(t-1)^{1/2m}} = (1 + \operatorname{ctg}^2\alpha)^{m+1/2} \times$$

$$\frac{1}{(t^2-1)^{1/2m+1/2}} \sum_{k=0}^{\infty} (-1)^k \frac{(m + {}^1/_2)_k (t-1)^k}{k!\,(t+1)^k} \operatorname{ctg}^{2k}\alpha$$

and integrating term by term, we obtain

$$P_\nu{}^m(\cos\alpha)\, P_\nu{}^m(-\cos\alpha) = (-1)^{m-1} \frac{2^m \Gamma(m + {}^1/_2)}{\pi\sqrt{\pi}} \times$$

$$\sin^{2m}\alpha \sin\pi\nu\, (1 + \operatorname{ctg}^2\alpha)^{m+1/2} \sum_{k=0}^{\infty} (-1)^k \frac{(m + {}^1/_2)_k}{k!} \operatorname{ctg}^{2k}\alpha \int_1^\infty \frac{P_\nu{}^m(t)\,(t-1)^k\, dt}{(t^2-1)^{1/2m+1/2}(t+1)^k}$$

If for the computation of the integral in the right-hand side of this equality we make use of the known formulas of the theory of hypergeometric functions [2], then we obtain formula (4.1) for $\nu = -{}^1/_2 + i\tau$. A particular case of this formula, corresponding to $m = 0$, has been given in [1].

*) This representation has been communicated to the author by N. N. Lebedev and I. P. Skal'skaia.

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ON THE USE OF RATIONAL APPROXIMATING FUNCTIONS

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Boundary value problems of the linear theory of viscoelasticity are solved using rational functions to approximate a function of the Poisson ratio. The problem of interpolation is solved for the class of rational fractions and the error of the approximation is estimated.

1. Suppose that a sufficiently smooth function $\varphi(\omega)$ of the real variable ω is to be approximated on the interval $a \leqslant \omega \leqslant b$ using another prescribed function, to a specified accuracy. The problem embraces that of interpolation, the latter consisting of finding the interpolation function $f_N(\omega)$ belonging to some class F and assuming, at the interpolation nodes, i.e. at certain prescribed points

$$\omega_0, \omega_1, \omega_2, \ldots, \omega_N \tag{1.1}$$

of the segment $[a, b]$, the same values as the function $\varphi(\omega)$, i.e.

$$f_N(\omega_0) = \varphi_0, \quad f_N(\omega_1) = \varphi_1, \ldots, \quad f_N(\omega_N) = \varphi \tag{1.2}$$

where

$$\varphi_n \equiv \varphi(\omega_n), \qquad n = 0, 1, \ldots, N \tag{1.3}$$

Depending on the class F, the interpolation problem may have an infinite number of solutions, or none. If polynomials of degree not greater than N are used to represent the function f_N, then the interpolation problem has a unique solution. In this case the polynomials are called the interpolation polynomials. Sometimes the properties of the function $\varphi(\omega)$ are such that it is more convenient to write the functions $f_N(\omega)$ in the form of rational fractions

$$f_N(\omega) = \sum_{i=0}^{M} p_i\omega^i / P_N, \qquad P_N = \sum_{i=1}^{N} q_i\omega^i \tag{1.4}$$

where p_i and q_i are constants. Obviously, the approximation (1.4) is more general than that employing the polynomials.